Seminar Geometrie und Topologie
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ALGEBRAIC SIMPLICITY OF PARTICULAR GROUPS OF HOMEOMORPHISMS

On the original paper [**1**] from R.D. Anderson

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Let $\mathbb{Q}$ be the set of rational numbers, $\mathbb{R}\backslash\mathbb{Q}$ the set of irrationals and $\mathcal{C}$ the Cantor set. We provide these sets with their induced topology from $\mathbb{R}$. We will see during this following developpement that these sets have some very nice 'homogeneity' properties that will let us claim this :

Let $X \in \{\mathbb{Q}, \mathbb{R}\backslash\mathbb{Q}, \mathcal{C}\}$ and

$$G = Homeo(X) = \{f : X \longrightarrow X \mid f \text{ is bijective, } f \quad \text{and} \quad f^{-1} \text{ are continuous } \}$$

the set of all homeomorphisms from X onto itself. We provide G with the composition law $\circ$, it is a group.

We shall show in this paper that the group of homeomorphisms of the topological spaces cited above is simple. More precisely, we shall see that :

Theorem Let h be a non trivial element of G, then, each element g of G is a product of 8 conjugates of h and h^{-1}.

The simplicity can be easily seen from such a property : let $H \trianglelefteq G$ $(H \neq \{id\})$, $h \in H\backslash\{id\}$ it follows that each conjugate $\mathfrak{K}(h)$ and $\mathfrak{K}(h^{-1})$ of h and h^{-1} is in H, and hence their product. From the Theorem above, the product of (8) conjugates of h and h^{-1} is in G, hence $G = H$. As each non trivial normal subgroup of G is itself, we conclude that G is simple.

To prove that theorem, we will list (without proof) some easy to see (or well known) properties of G and X that will be used through the proof. From now on, X and G are defined as above, $\mathcal{O}(X)$ denote the topology on X. An element g is said to be *supported* on a set k if $g = id$ outside of k. G^0 denotes the subset of G of all the elements that are the identity in some non empty open subsets of X.

Let $K(X)$ be the collection of all **clopen** (closed-and-open) non empty proper subsets of X; to easily see such subsets, we can consider in the case $X = \mathbb{Q}$, the sets

$$A = \{r \in \mathbb{Q}, -\sqrt{2} \leq r \leq \sqrt{2}\} =]-\sqrt{2}, \sqrt{2}[\cap\mathbb{Q} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$$

As $\mathbb{Q}$ inherits its topology from $\mathbb{R}$, if U is open (or closed) in $\mathbb{R}$, so is $U \cap \mathbb{Q}$ in $\mathbb{Q}$. One can easily see then, that A is open ($]-\sqrt{2}, \sqrt{2}[$ is open in $\mathbb{R}$) and closed ($[\sqrt{2}, \sqrt{2}]$ is closed in $\mathbb{R}$) in $\mathbb{Q}$.

Properties Let X be as defined above, we have the following properties :

(1) For $U \in \mathcal{O}(X)$, there exists a $k \in K(X)$ s.t. $k \subset U$

(2) For $k \in K(X)$, there exists a countably infinite collection of disjoint sets $(k_i)_{i\in\mathbb{N}} \in K(X)$, α_1,$\alpha_2 \in G^0$ s.t.

(a)
$$\bigcup_{i\in\mathbb{N}} k_i \subset k$$

(b)
$$\alpha_1(k_i) = k_{i+1} \qquad \text{for each } i$$

(c)
$$\alpha_1 \,|k_0 = \alpha_2 \,|k_0$$

and for $i > 0$

$$\alpha_2 \,|k_{2i} = \alpha_1^{-2} \,|k_{2i} \quad \text{and} \quad \alpha_2 \,|k_{2i-1} = \alpha_1^{2} \,|k_{2i-1}$$

(3) if for each i, $\phi_i \in G^0$ is supported on k_i, there exists $\phi \in G^0$ s.t.

$$\phi \text{ is supported on } \bigcup_{i\in\mathbb{N}} k_i \quad \text{ and } \quad \forall i \quad \phi\,|k_i = \phi_i\,|k_i$$

(4) G is transitive i.e. for any $k, k' \in K(X)$ there exists an $t \in G$ s.t. $t(k') = k$

(5) let A be a subset of X and $g \in G^0$ supported on A, then for any $\psi \in G$

$$\psi^{-1}g\psi \quad \text{ is supported on } \psi^{-1}(A)$$

If $\psi^{-1}(A) \cap A = \emptyset$

$$f = \psi^{-1}g^{-1}\psi g \quad \text{ is supported on } A \cup \psi^{-1}(A)$$

while
$$f|A = g|A \quad \text{and} \quad f|\psi^{-1}(A) = \psi^{-1}g\psi|\psi^{-1}(A)$$

We note that most of those properties come from the fact that $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$ and $\mathcal{C}$ are zero-dimensional (with respect to the inductive dimension [**4**]) Hausdorff spaces (T_2), they are then totally disconnected and have a base consisting of clopen subsets (each two distinct points can be separated by two disjoint clopen sets); for further readings about these spaces properties, refer to [**2**] and [**3**].

Proof. We will first show that every g_0 in G^0 is a product of 4 conjugates of h and h^{-1}. At the end, we will see that this implies the theorem.

Let $g_0 \in G^0$, g_0 is supported on some $k \in K(X)$. Let $k_0 \in K(X)$, from **(4)** there exists an $\alpha \in G$ such that $\alpha(k) = k_0$. By posing

$$g' := \alpha g_0 \alpha^{-1} \tag{$*$}$$

g' is supported on $\alpha(k) = k_0$ by **(5)** and g_0 is a conjugate of g'.

As $h \in G\backslash\{id\}$, there exist a p in X such that $h(p) \neq p$. As X is T_2, there exists two clopen neighbourhoods $\mathcal{V}_{h(p)}$ and $\mathcal{V}_p$ such that

$$\mathcal{V}_{h(p)} \cap \mathcal{V}_p = \emptyset$$

Since h and h^{-1} are continuous functions, it follows that $h(k)$ and $h^{-1}(k)$ are clopen sets. By choosing $h(k) = \mathcal{V}_{h(p)}$ and $\mathcal{V}_p = k$ we have

$$\begin{cases} h(k) \cap k = \emptyset \\ h^{-1}(k) \cap k = \emptyset \end{cases} \quad \text{and hence} \quad k \cap [\ h(k) \cup h^{-1}(k)\ ] = \emptyset$$

From **(2)**, there exists a disjoint collection $(k_i)_{i\geq 0} \subset K(X)$ and α_1, α_2 supported on k satisfying properties **(a)**, **(b)** and **(c)**.

Let $\phi_i = \alpha_1^i g' \alpha_1^{-i}$, $i \geq 0$

for $i = 0$, $\phi_0 = g'$ is supported on k_0
By induction on $i > 0$, suppose that $\alpha_1^i g' \alpha_1^{-i}$ is supported on k_i,

$$\alpha_1^{i+1} g' \alpha_1^{-i-1} = \alpha_1 g' \alpha_1 \text{ is supported on } \alpha_1(k_i) = k_{i+1} \text{ (by } \mathbf{(5)} \text{)}$$

Hence, ϕ_i is supported on k_i for each i. By **(3)**, there exists ϕ supported on $\bigcup k_i$ such that

$$\phi\ |k_0 = g'\ |k_0 \quad \text{and} \quad \forall i > 0 \quad \phi\ |k_i = \phi_i\ |k_i$$

Let us consider $f := h^{-1}\phi^{-1}h\phi$ and $Y := [\ \bigcup k_i\ ] \cup [\ h^{-1}(\bigcup k_i)\ ]$.

As ϕ is supported on $\bigcup k_i$, it follows from **(5)** that

$$h^{-1}\phi h \text{ is supported on } h^{-1}(\bigcup_{i\geq 0} k_i) = \bigcup_{i\geq 0} h^{-1}(k_i)$$

As $h^{-1}(k) \cap k = \emptyset$, from the second part of property **(5)**, f is supported on Y while

$$f\ |k = \phi\ |k \quad \text{and} \quad f\ |h^{-1}(k) = h^{-1}\phi h\ |h^{-1}(k)$$

Now, consider $\rho := h^{-1}\alpha_2 h\alpha_1^{-1}$ and $\omega := \rho^{-1}f^{-1}\rho f$.

$$\rho\ |k = h^{-1}\alpha_2 h\alpha_1^{-1}\ |k = h^{-1}\alpha_2 h\ |k\ \alpha_1^{-1}\ |k = \alpha_1^{-1}\ |k$$

(as $h^{-1}\alpha_2 h$ is supported on $h(k)$ and $k \cap h(k) = \emptyset$)

$$\rho\ |h^{-1}(k) = h^{-1}\alpha_2 h\alpha_1^{-1}\ |h^{-1}(k) = h^{-1}\alpha_2 h\ |h^{-1}(k)\ \alpha_1^{-1}\ |h^{-1}(k) = h^{-1}\alpha_2 h\ |h^{-1}(k)$$

(by the same argument, with α_1 supported on k)

We can already see that

$$\omega = \rho^{-1}f^{-1}\rho f = \rho^{-1}\phi^{-1}h^{-1}\phi h\rho h^{-1}\phi^{-1}h\phi = (\rho^{-1}\phi^{-1}h^{-1}\phi h\rho)(\rho^{-1}h\rho)(h^{-1})(\phi^{-1}h\phi)$$

is the product of 4 conjugates of h and h^{-1}. We wish to check that $\omega = g'$; to do so we need merely to track down the action of ω on X.

Let us have a look on $\rho(Y)$:

$$\rho^{-1}(Y) = \alpha_1 h^{-1}\alpha_2^{-1}h(Y) = \alpha_1 h^{-1}\alpha_2^{-1}[\ \bigcup_{i\geq 0}(h(k_i) \cup k_i)\ ]$$

We need here to evaluate α_1 and α_2 at elements of the set $\bigcup(h(k_i) \cup k_i)$. As α_2 is supported on k and knowing that $k \cap h(k) = \emptyset$, we can already see that the action of α_2 on $h(k_i)$ is trivial for each $i > 0$

$$\rho^{-1}(Y) \subset \alpha_1 h^{-1}[\ \bigcup_{i\geq 0}(h(k_i) \cup \alpha_2^{-1}(k_i))\ ] \subset \alpha_1[\ \bigcup_{i\geq 0}(k_i \cup h^{-1}(k_i))$$

Here again, as k and $h^{-1}(k)$ are disjoint, the action of α_1 on $h^{-1}(k_i)$ is trivial for each i and hence

$$\rho^{-1}(Y) \subset \bigcup_{i\geq 0}(\alpha_1(k_i) \cup h^{-1}(k_i)) \subset Y$$

We have seen that f is supported on Y and as $Y \subset \rho(Y)$, $\omega = \rho^{-1}f^{-1}\rho f$ is supported on Y while each of f and ρ is the product of two elements of G^0, one supported on k and the other on $h^{-1}(k)$. Therefore, to study the action of ω we may consider its effect on k and $h^{-1}(k)$ separately.

♦ Consider the restriction of ω on k :

$$\omega\ |k = \rho^{-1} f^{-1} \rho f\ |k = \alpha_1 \phi^{-1} \alpha_1^{-1} \phi\ |k$$

$\omega\ |k$ is supported on Y, it is supported on $\bigcup k_i$ for each $i \geq 0$.

for $i = 0$

$$\omega\ |k_0 = \alpha_1 \phi^{-1} \alpha_1^{-1} \phi\ |k_0 = (\alpha_1 \phi^{-1} \alpha_1^{-1}\ |k_0)\ (g'\ |k_0) = (\alpha_1 g'^{-1} \alpha_1^{-1}\ |k_0)\ (g'\ |k_0)$$

$$= (\phi_1^{-1}\ |k_0)\ (g'\ |k_0) = g'\ |k_0$$

for $i > 0$

$$\omega\ |k_i = \alpha_1 \phi^{-1} \alpha_1^{-1} \phi\ |k_i = (\alpha_1 \phi^{-1} \alpha_1^{-1}\ |k_i)\ (\phi\ |k_i) = (\alpha_1 \phi^{-1}\ |k_{i-1})\ (\alpha_1^{-1} \phi\ |k_i)$$

$$= \alpha_1 (\alpha_1^{i-1} g'^{-1} \alpha^{-i+1}) \alpha_1^{-1} (\phi\ |k_i) = (\alpha_1^{i} g'^{-1} \alpha^{-i})\ (\phi\ |k_i)$$

$$= \phi_i^{-1} \phi_i\ |k_i = id$$

Thus, $\omega\ |k$ is $g'\ |k$

♦ Consider now the restriction of ω on $h^{-1}(k)$:

$$\omega\ |h^{-1}(k) = \rho^{-1} f^{-1} \rho f\ |h^{-1}(k) = (h^{-1} \alpha_2^{-1} h)(h^{-1} \phi h)(h^{-1} \alpha_2 h)(h^{-1} \alpha_2 h)\ |h^{-1}(k)$$

$$= h^{-1} \alpha_2^{-1} \phi \alpha_2 \phi^{-1} h\ |h^{-1}(k)$$

Here, to see how ω looks like on $h^{-1}(k)$, we will first have a look on ($\alpha_2^{-1} \phi \alpha_2 \phi^{-1}$) on k.

for $i = 0$

$$\alpha_2^{-1} \phi \alpha_2 \phi^{-1}\ |k_0 = (\alpha_2^{-1} \phi \alpha_2\ |k_0)\ (\phi^{-1}\ |k_0) = (\alpha_1^{-1} \phi \alpha_1\ |k_0)\ (g'^{-1}\ |k_0)$$

$$= (\alpha_1^{-1} \phi\ |k_1)(\alpha_1 g'^{-1}\ |k_0) = (\alpha_1^{-1}\ |k_1)(\alpha_1 g' \alpha_1^{-1} \alpha_1 g'^{-1}\ |k_0)$$

$$= (\alpha_1^{-1}\ |k_1)(\alpha_1\ |k_0) = (\alpha_1^{-1} \alpha_1)\ |k_0 = id$$

for $i > 0$

$$\alpha_2^{-1}\phi\alpha_2\phi^{-1}\,|k_{2i} = (\alpha_2^{-1}\phi\alpha_2\,|k_{2i})\,(\phi^{-1}\,|k_{2i}) = (\alpha_2^{-1}\phi\alpha_1^{-2}\,|k_{2i})\,(\alpha_1^{2i}g'^{-1}\alpha_1^{-2i}\,|k_{2i})$$

$$= (\alpha_2\phi\,|k_{2i-2})\,(\alpha_1^{2i-2}g'^{-1}\alpha_1^{-2i}\,|k_{2i}) = (\alpha_2^{-1}\,|k_{2i-2})\,(\alpha_1^{2i-2}g'\alpha_1^{-2i+2}\alpha_1^{2i-2}g'^{-1}\alpha_1^{-2i}\,|k_{2i})$$

$$= (\alpha_1^{2}\,|k_{2i-2})\,(\alpha_1^{-2}\,|k_{2i}) = (\alpha_1^{-2}\alpha_1^{2}\,|k_{2i}) = id$$

And

$$\alpha_2^{-1}\phi\alpha_2\phi^{-1}\,|k_{2i-1} = (\alpha_2^{-1}\phi\alpha_2\,|k_{2i-1})\,(\phi^{-1})\,|k_{2i-1} = (\alpha_2^{-1}\phi\alpha_1^{2}\,|k_{2i-1})\,(\alpha_1^{2i-1}g'^{-1}\alpha_1^{-2i+1}\,|k_{2i-1})$$

$$= (\alpha_2\phi\,|k_{2i+1})\,(\alpha_1^{2i+1}g'^{-1}\alpha_1^{-2i+1}\,|k_{2i-1}) = (\alpha_2^{-1}\,|k_{2i+1})\,(\alpha_1^{2i+1}g'\alpha_1^{-2i-1}\alpha_1^{2i+1}g'^{-1}\alpha_1^{-2i+1}\,|k_{2i-1})$$

$$= (\alpha_1^{-2}\,|k_{2i+1})\,(\alpha_1^{2}\,|k_{2i-1}) = (\alpha_1^{-2}\alpha_1^{2}\,|k_{2i-1}) = id$$

Thus, $\alpha_2^{-1}\phi\alpha_2\phi^{-1}\,|k$ is the identity, and hence

$$\omega\,|h^{-1}(k) = h^{-1}\alpha_2^{-1}\phi\alpha_2\phi^{-1}h\,|h^{-1}(k) = (h^{-1}\alpha_2^{-1}\phi\alpha_2\phi^{-1}\,|k)\,(h\,|h^{-1}(k))$$

$$= (h^{-1}\,|k)\,(h\,|h^{-1}(k)) = (h^{-1}h)\,|h^{-1}(k) = id$$

This shows that ω is supported on k and that $g' = \omega$ is a product of 4 conjugates of h and h^{-1}, from $(*)$ so is g_0.

Finally, we have shown that each element g_0 of G^0 is a product of 4 conjugates of h and h^{-1}.

Let's take an arbitary (non trivial) g from G and a special clopen set k this time, such that

$$\begin{cases} g(k) \cap k &= \emptyset \\ g(k) \cup k &\neq X \end{cases}$$

Such a k exist by looking at the followings arguments : as $g \neq id$, $\exists p \in X$ such that $g(p) \neq p$. Again from the T_2 space propertie of X, there exist twe clopen neighbourhoods $\mathcal{V}_{g(p)} := U$ and $\mathcal{V}_p := V$ such that

$$U \cap V = \emptyset$$

By considering $k := U \cap g^{-1}(V)$; we have clearly that $g(k) \cap k = \emptyset$ and, by taking k smaller, that $g(k) \cup k \neq X$ also.

Now consider $k' = g(k) \cup k \in K(X)$ and define λ such that

$$\begin{cases} \lambda\,|k &= g\,|k \\ \lambda\,|g(k) &= g^{-1}\,|g(k) \\ \lambda\,|X \backslash k &= id\,|X \backslash k \end{cases}$$

λ is in G^0 and so is $\lambda^{-1}g$. As elements of G^0, we have shown that each of them is a product of 4 conjugates of h and h^{-1}, hence

$$g = (\lambda)(\lambda^{-1}g) \text{ is a product of 8 conjugates of } h \text{ and } h^{-1}$$

$\square$

Remark The theorem can be slightly sharpened : R.D Anderson proves in his paper that an element of G can be written as a product of 6 conjugates of h and h^{-1}. He also extend his theorem to the group of **orientation-preserving homeomorphisms** of the 2-Sphere (S^2), the 3-Sphere (S^3) and the "Swiss cheese" (universal plane curves, a famous example would be *the Sierpinski carpet*).

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